

ON THE DIFFERENTIABILITY OF THE SOLUTIONS OF QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS⁽¹⁾

BY

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Introduction. In our thesis [2, Appendix, pp. 106–118] (cf. also [3]) we proved the infinite differentiability of the solutions of quasilinear partial differential equations of the form:

$$(1) \quad \sum_{j=1}^{\mu} a_j(x, N_1(D)f, \dots, N_\nu(D)f) M_j(D)f = g(x, N_1(D)f, \dots, N_\nu(D)f).$$

Our hypotheses were: (1) The differential operator

$$\sum_{j=1}^{\mu} a_j(x_0, N_1(D)f(x_0), \dots, N_\nu(D)f(x_0)) M_j(D)$$

is, for every x_0 , hypoelliptic and stronger than each $M_j(D)$. (2) Each $N_k(D)$ is strictly weaker than all $M_j(D)$ together (cf. [2, p. 116, d]). (3) The functions $a_j(x, t_1, \dots, t_\nu)$ and $g(x, t_1, \dots, t_\nu)$ are infinitely differentiable. In this paper we want to outline a different proof which is more elementary than the original one, in the sense that it utilizes neither the Sobolev nor the Gagliardo-Nirenberg estimates but only rather straightforward estimates obtained from the Fourier transformation in L^2 . A severe disadvantage, on the other hand, is the fact that it requires rather strong a priori differentiability of the solutions.

1. The Schauder algebra. Let s be any real number. Put

$$\|f\|_s = \left(\int (1 + |\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}, \quad f \in C_c^\infty$$

($\hat{f}(\xi)$ is the Fourier transform of $f=f(x)$, C_c^∞ is the set of infinitely differentiable functions with compact supports) and let H^s be the completion of C_c^∞ in this norm.

LEMMA 1. *If $s > n/2$, then there exists a constant K such that*

$$(2) \quad \|fg\|_s \leq K \|f\|_s \|g\|_s, \quad f \in C_c^\infty, g \in C_c^\infty.$$

Proof. Put $h=fg$. Now $\hat{h}(\xi) = \int \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta$. Hence

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$$(3) \quad \|h\|_s^2 = \iint (1 + |\xi|)^{2s} \hat{f}(\xi - \eta) \hat{g}(\eta) (\hat{h}(\xi))^* d\xi d\eta.$$

By Schwarz's inequality, we then obtain

$$(4) \quad \begin{aligned} \|h\|_s^2 &\leq \left(\int (1 + |\xi - \eta|)^{2s} |\hat{f}(\xi - \eta)|^2 d\xi \right)^{1/2} \\ &\times \left(\int (1 + |\eta|)^{2s} |\hat{g}(\eta)|^2 d\eta \right)^{1/2} \\ &\times \left(\iint \left(\frac{1 + |\xi|}{(1 + |\xi - \eta|)(1 + |\eta|)} \right)^{2s} (1 + |\xi|)^{2s} |\hat{h}(\xi)|^2 d\xi d\eta \right)^{1/2}. \end{aligned}$$

The first two factors on the right hand side of (4) are $\|f\|_s$ and $\|g\|_s$, respectively. Now

$$\begin{aligned} \left(\frac{1 + |\xi|}{(1 + |\xi - \eta|)(1 + |\eta|)} \right)^{2s} &\leq \left(\frac{(1 + |\xi - \eta|) + (1 + |\eta|)}{(1 + |\xi - \eta|)(1 + |\eta|)} \right)^{2s} \\ &= \left(\frac{1}{1 + |\xi - \eta|} + \frac{1}{1 + |\eta|} \right)^{2s} \\ &\leq 2^{2s-1} \left(\frac{1}{(1 + |\xi - \eta|)^{2s}} + \frac{1}{(1 + |\eta|)^{2s}} \right). \end{aligned}$$

It follows that the square of the third factor is

$$\begin{aligned} &\leq 2^{2s-1} \int \left(\int \frac{d\eta}{(1 + |\xi - \eta|)^{2s}} + \int \frac{d\eta}{(1 + |\eta|)^{2s}} \right) (1 + |\xi|)^{2s} |\hat{h}(\xi)|^2 d\xi \\ &= 2^{2s} \int \frac{d\eta}{(1 + |\eta|)^{2s}} \|h\|_s^2 = K^2 \|h\|_s^2 \end{aligned}$$

with $K = 2^s (\int (1 + |\eta|)^{-2s} d\eta)^{1/2}$, which is finite in view of the assumption that $s > n/2$. Hence $\|h\|_s^2 \leq \|f\|_s \|g\|_s K \|h\|_s$, and (2) follows.

COROLLARY (SCHAUDER). *If $s > n/2$, then H^s is a Banach algebra (the Schauder algebra).*

REMARK. In the above reasoning one can as well replace $1 + |\xi|$ by an arbitrary "weight function" $E(\xi)$ such that

$$(5) \quad \frac{E(\xi)}{E(\xi - \eta)E(\eta)} \leq c \left(\frac{1}{(1 + |\xi - \eta|)^\delta} + \frac{1}{(1 + |\eta|)^\delta} \right)$$

for some $\delta > 0$. In particular, (5) is fulfilled (Malgrange) if $E(\xi) = 1 + |P(\xi)|$ where $P(\xi)$ is a hypoelliptic polynomial. This can easily be seen by making use of Leibniz's formula in its "tensorial" form [1, p. 292].

LEMMA 2. If $s > n/2 + 1$, then there exists a constant K' such that

$$(6) \quad \|fg\|_s \leq \sup |g| \|f\|_s + K' \|g\|_s \|f\|_{s-1}, \quad f \in C_c^\infty, g \in C_c^\infty.$$

Proof. Write (3) as

$$\begin{aligned} \|h\|_s^2 &= \iint (1 + |\xi|)^s (1 + |\xi - \eta|)^s \hat{f}(\xi - \eta) \hat{g}(\eta) (\hat{h}(\xi))^* d\xi d\eta \\ &\quad + \iint (1 + |\xi|)^s ((1 + |\xi|)^s - (1 + |\xi - \eta|)^s) \hat{f}(\xi - \eta) \hat{g}(\eta) (\hat{h}(\xi))^* d\xi d\eta \\ &= \text{I} + \text{II}. \end{aligned}$$

Since $\text{I} = (g(1 + |D|)^s f, (1 + |D|)^s h)$, we obtain

$$|\text{I}| \leq \|g(1 + |D|)^s f\|_0 \|(1 + |D|)^s h\|_0 \leq \sup |g| \|f\|_s \|h\|_s,$$

where $|D|$ stands for multiplication of the Fourier transform by $|\xi|$. It remains to estimate II. We observe that

$$|(1 + |\xi|)^s - (1 + |\xi - \eta|)^s| \leq s((1 + |\xi|)^{s-1} + (1 + |\xi - \eta|)^{s-1}) |\eta|.$$

Hence we have to consider:

$$\text{II}_a = s \iint (1 + |\xi|)^s (1 + |\xi - \eta|)^{s-1} (1 + |\eta|) |\hat{f}(\xi - \eta)| |\hat{g}(\eta)| |\hat{h}(\xi)| d\xi d\eta$$

and

$$\text{II}_b = s \iint (1 + |\xi|)^s (1 + |\xi|)^{s-1} (1 + |\eta|) |\hat{f}(\xi - \eta)| |\hat{g}(\eta)| |\hat{h}(\xi)| d\xi d\eta$$

where

$$|\text{II}| \leq |\text{II}_a| + |\text{II}_b|.$$

As in the proof of Lemma 1, we obtain by Schwarz's inequality:

$$\begin{aligned} |\text{II}_a| &\leq s \left(\int (1 + |\xi - \eta|)^{2(s-1)} |\hat{f}(\xi - \eta)|^2 d\xi \right)^{1/2} \\ &\quad \times \left(\int (1 + |\eta|)^{2s} |\hat{g}(\eta)|^2 d\eta \right)^{1/2} \\ &\quad \times \left(\int \frac{d\eta}{(1 + |\eta|)^{2(s-1)}} \right)^{1/2} \times \left(\int (1 + |\xi|)^{2s} |\hat{h}(\xi)|^2 d\xi \right)^{1/2} \\ &= s \left(\int \frac{d\eta}{(1 + |\eta|)^{2(s-1)}} \right)^{1/2} \|f\|_{s-1} \|g\|_s \|h\|_s \end{aligned}$$

and

$$\begin{aligned}
|II_b| &\leq s \left(\int (1 + |\xi - \eta|)^{2(s-1)} |\hat{f}(\xi - \eta)|^2 d\xi \right)^{1/2} \\
&\quad \times \left(\int (1 + |\eta|)^{2s} |\hat{g}(\eta)|^2 d\eta \right)^{1/2} \\
&\quad \times \left(\iint \left(\frac{1 + |\xi|}{(1 + |\xi - \eta|)(1 + |\eta|)} \right)^{2(s-1)} (1 + |\xi|)^{2s} |\hat{h}(\xi)|^2 d\xi d\eta \right)^{1/2} \\
&\leq s 2^{s-1} \left(\int \frac{d\eta}{(1 + |\eta|)^{2(s-1)}} \right)^{1/2} \|f\|_{s-1} \|g\|_s \|h\|_s.
\end{aligned}$$

Putting together the estimates obtained for I, II_a, II_b, clearly (6) follows with

$$K' = s(2^{s-1} + 1) \left(\int \frac{d\eta}{(1 + |\eta|)^{2(s-1)}} \right)^{1/2}.$$

2. **The linear case.** We consider now (cf. [4]) differential operators of the form:

$$(7) \quad P = P(x, D) = \sum_{j=1}^{\mu} a_j(x) M_j(D)$$

where we assume that (1) $M = M(D) = P(0, D)$ is stronger than each M_j ,
 (2) $a_j \in H^s$ for some $s > n/2 + 1$ and the quantity

$$\zeta = \sum_{j=1}^{\mu} \sup |a_j(x) - a_j(0)|$$

is less than a positive constant ζ_0 to be determined, depending on M and M_j only. We claim that the following inequality holds:

$$(8) \quad \|Mf\|_s \leq C(\|Pf\|_s + \|Mf\|_{s-1}), \quad f \in C_v^\infty,$$

where C_v^∞ is the subset of C^∞ of functions f in C^∞ such that the support of f is contained in the unit sphere U . Put

$$b_j(x) = w(x)(a_j(x) - a_j(0))$$

where w is in C_c^∞ and equals 1 in a neighborhood of U . By the triangle inequality we then obtain

$$\|Mf\|_s \leq \|Pf\|_s + \sum \|b_j M_j f\|_s$$

and, by Lemma 2,

$$\leq \|Pf\|_s + \sum \sup |b_j| \|M_j f\|_s + K' \|b_j\|_s \|M_j f\|_{s-1}.$$

Now there exist (cf. [2, Proposition 2, p. 43]) constants $\gamma, \gamma', \gamma''$, of which γ does not depend on s , such that

$$\begin{aligned} \|M_j f\|_s &\leq \gamma \|Mf\|_s + \gamma' \|Mf\|_{s-1}, \\ \|M_j f\|_{s-1} &\leq \gamma'' \|Mf\|_{s-1}, \end{aligned} \quad f \in C_U^\infty.$$

Hence

$$\|Mf\|_s \leq \|Pf\|_s + \zeta \gamma \|Mf\|_s + \zeta \gamma' \|Mf\|_{s-1} + Z \gamma'' K' \|Mf\|_{s-1},$$

with $Z = \sum \|b_j\|_s$, so that (8) follows if we choose $\zeta_0 = 1/\gamma$.

From (8) we now obtain, as usual, the following

THEOREM 1. *Suppose that (1) and (2) hold and that moreover $a_j \in H^{s+1}$ with $s > n/2 + 1$. Suppose that the support of f is contained in U and that $Mf \in H^s$ and $Pf \in H^{s+1}$. Then $Mf \in H^{s+1}$.*

Proof. Apply (8) to $(f_h - f)/|h|$ where $f_h = f_h(x) = f(x+h)$. In view of Lemma 1 the right hand side will remain bounded as h tends to 0 and hence also the left hand side. A familiar weak compactness argument then gives $Mf \in H^{s+1}$.

Assume now that M is hypoelliptic. Then there is a number $d > 0$ such that

$$|M_j^\alpha(\xi)| \leq C(1 + |\xi|)^{-d}(1 + |M(\xi)|), \quad \alpha \neq 0,$$

so that $Mf \in H^s$ implies $M_j^\alpha f \in H^{s+d}$, $\alpha \neq 0$, provided f has compact support. Theorem 1 can now be "localized":

THEOREM 2. *Suppose that the hypotheses of Theorem 1 and the foregoing assumptions are fulfilled, with $s > n/2 + 2 - d$. If $Mf \in H_{loc}^s(U)$ and $Pf \in H_{loc}^{s+1}(U)$, then $Mf \in H_{loc}^{s+1}(U)$.*

(Here $H_{loc}^s(U)$ is the set of functions f in U such that $\phi f \in H^s$ for every $\phi \in C_U^\infty$.)

Proof. Cf. e.g. [4] for details.

REMARK. Suppose next that P is of the form (7) and such that (1') $P(x_0, D)$ is stronger than each $M_j(D)$ for every fixed x_0 , (2') $a_j \in H^{s+1}$ for some $s > n/2 + 2 - d$. Let M be some fixed $P(x_0, D)$. Then we may conclude that $Mf \in H_{loc}^s(\emptyset)$ and $Pf \in H_{loc}^{s+1}(\emptyset)$ imply $Mf \in H_{loc}^{s+1}(\emptyset)$, for every open set $\emptyset$ in R^n . In fact it is apparently sufficient to establish this fact when $\emptyset$ is a small neighborhood of $x_0 = 0$, and in this case everything follows from Theorem 2, for P equals in the vicinity of 0 to some operator of the form (7) satisfying (1), (2).

3. The quasilinear case. We turn now to the quasilinear equation (1). Our hypotheses are the hypotheses (1), (2), (3) of the introduction. Let M be some fixed operator equally strong as all M_j together. Then we have the following

THEOREM 3. *There is a number σ_0 such that if f satisfies (1) and if $Mf \in H_{loc}^s(\emptyset)$ for some $s > \sigma_0$, $\emptyset$ being an open set of R^n , then $f \in C^\infty(\emptyset)$.*

For the proof we need the following

LEMMA 3. *There exists a number σ with the following properties. Let $a(x, t_1, \dots, t_\nu)$ be infinitely differentiable in $R^n \times C^\nu$ and suppose that all derivatives are bounded in every set of the form*

$$|t_1| \leq W, \dots, |t_\nu| \leq W, \quad x \text{ arbitrary.}$$

Suppose further that $a(x, 0, \dots, 0) \in H^s$ for some $s > \sigma$. Then $\phi_1 \in H^s, \dots, \phi_\nu \in H^s$ imply $a(x, \phi_1, \dots, \phi_\nu) \in H^s$.

Let us assume the lemma for a moment. Then Theorem 3 follows at once from Theorem 2, in the form given to it in the Remark following it. In fact, if $Mf \in H_{loc}^s(\emptyset)$, then, for some $\epsilon > 0$, $N_\epsilon f \in H_{loc}^{s+\epsilon}(\emptyset)$ and hence in virtue of Lemma 3, if $s > \sigma - \epsilon$, $a_j(x, N_1 f, \dots, N_\nu f)$ and $g(x, N_1 f, \dots, N_\nu f) \in H_{loc}^{s+\epsilon}(\emptyset)$, so that in view of Theorem 2, if further $s + \epsilon - 1 > n/2 + 2 - d$, $Mf \in H_{loc}^{s+\epsilon}(\emptyset)$. Repeating this argument l times, one obtains $Mf \in H_{loc}^{s+l}(\emptyset)$ and, letting l tend to infinity, $Mf \in C^\infty(\emptyset)$ and hence $f \in C^\infty(\emptyset)$. This proves the theorem, with

$$\sigma_0 = \sup(\sigma - \epsilon, n/2 + 3 - d - \epsilon).$$

It remains to prove Lemma 3.

Proof of Lemma 3. Introducing $\operatorname{Re} t_k$ and $\operatorname{Im} t_k$ as new variables, we may as well assume that $t_1, \dots, t_\nu$ are real variables. Also it is no restriction to assume that $a(x, t_1, \dots, t_\nu)$ is periodic in each t_k with period, say, 2π . Expand $a(x, t_1, \dots, t_\nu)$ in trigonometric series,

$$a(x, t_1, \dots, t_\nu) = \sum a_{l_1 \dots l_\nu}(x) e^{i(l_1 t_1 + \dots + l_\nu t_\nu)}$$

where $l_1, \dots, l_\nu$ are integers. We obtain

$$a(x, \phi_1, \dots, \phi_\nu) = a(x, 0, \dots, 0) + \sum a_{l_1 \dots l_\nu}(x) (e^{i(l_1 \phi_1 + \dots + l_\nu \phi_\nu)} - 1).$$

The first term is in H^s by hypothesis. Also the coefficients $a_{l_1 \dots l_\nu}(x)$ and all their derivatives tend to zero rapidly, i.e., faster than any expression of the form $l/(1 + |l_1| + \dots + |l_\nu|)^A$. So everything will follow if we can show that

$$\psi = e^{i(l_1 \phi_1 + \dots + l_\nu \phi_\nu)} - 1$$

is in H^s and moreover subject to a majorization of the form

$$(9) \quad \|\psi\|_s \leq C(1 + |l_1| + \dots + |l_\nu|)^B.$$

Clearly $\psi \in H^t$ for some $t > n/2$, provided $s > 0$, and 0 is chosen so that all derivatives of order $\leq t$ will be continuous and bounded, which is possible in view of a (weak) form of Sobolev's lemma. Differentiating we get

$$D_\alpha \psi = i(\psi + 1) \sum l_k D_\alpha \phi_k, \quad |\alpha| = 1,$$

so that, by the Corollary of Lemma 1, $D_\alpha \psi \in H^{\operatorname{int}(t, s-1)}$. This improves the

regularity of ψ from t to $\inf(t+1, s)$ so finally $\psi \in H^s$. The same argument (we omit the details!) proves also the inequality (9).

REMARK. Actually, as follows from [2, Proposition 1, p. 116], one can take $\sigma = n/2$. For our purpose the present weaker but more elementary statement is of course sufficient.

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